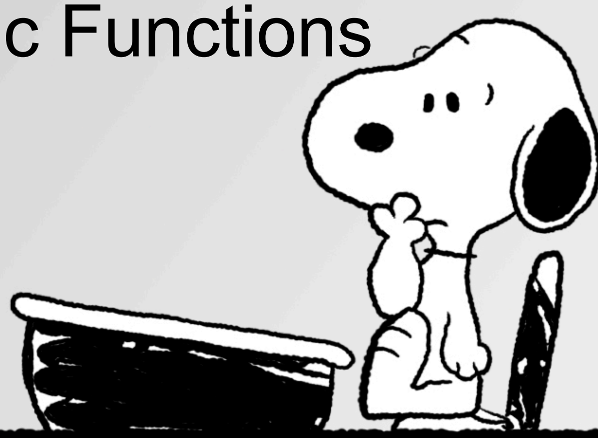
LG #5

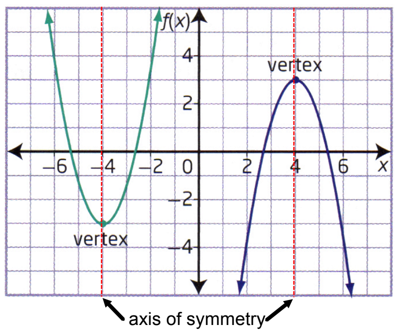
Quadratic Functions



Agenda:

Topic 1 Quadratics

A quadratic is a function where the x value is squared. The simplest quadratic is $f(x)=x$



- The graph of a quadratic function is a parabola
- The lowest or highest point on the graph is the vertex
- The axis of symmetry divides the graph into mirror images and its' equation corresponds to the x-coordinate of the vertex

Quadratic Functions can be written in vertex form or standard form. Vertex form is useful for graphing.

vertex form

$$f(x) = a(x-p)^2 + q$$

stretch/shrink
opens up or down

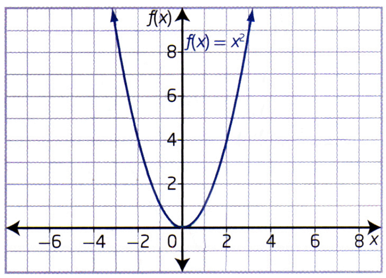
vertex (p, q)

standard form

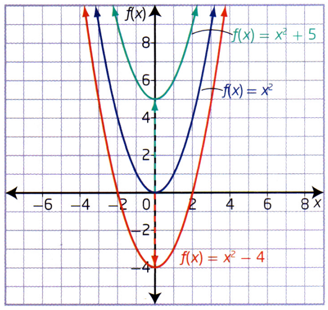
$$f(x) = ax^2 + bx + c$$

The effects of $a, p \,\&\, q$ on the function

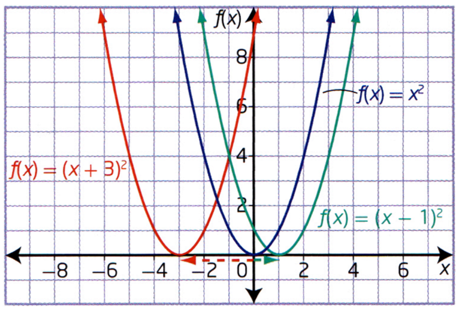
1. Basic Parabola $y = x^2$



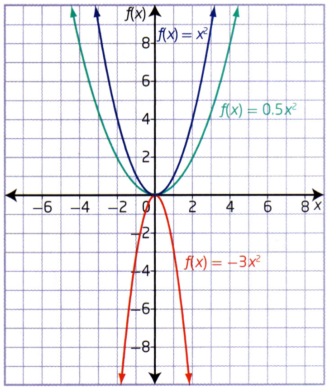
2. The effect of q (vertical shift) $y = x^2 + q$



3. The effect of p (horizontal shift) $y = (x - p)^2$



4. The effect of a (stretch & shrink) $y = ax^2$



The value of a tells you 2 things:

a) graph opens up or down

- $a > 0$ opens up
- $a < 0$ opens down

b) stretch or shrink

- $-1 < a < 1$ graph is wider than $y = x^2$
- $a > 1$ or $a < 1$ graph is more narrow than $y = x^2$

Example 1

Sketch Graphs of Quadratic Functions in Vertex Form

Example: Determine the following for a function:

- the vertex
- the domain and range
- the direction of the opening
- the equation of the axis of symmetry

Then, sketch the graph

$$y = 2(x+1)^2 - 3$$

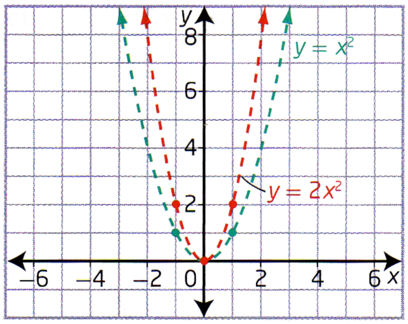
$a = 2$ $p = -1$ $q = -3$

vertex: (-1, -3)
opens: up (a> 0) and is narrower
than $y = x^2$ (a> 1)
domain: $\{x | x \in R\}$ or x = all real numbers
(for all parabolas)
range: $\{y | y \geq -3, y \in R\}$ or y≥ -3 (q = -3)
axis of symmetry: x = -1 (p = -1)

Example 1 (cont.)

Method 1: Sketch using Transformations

1. Start with the graph of $y = x^2$
   - Use the points (0, 0), (1, 1), (-1, 1), (2, 4) and (-2, 4) to graph $y = x^2$



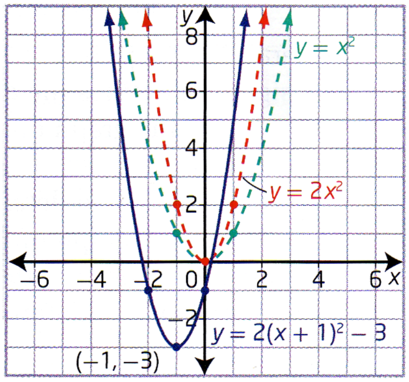
- Apply the change in width (a) first:

(0, 0) → (0, 0)
(1, 1) → (1, 2)
(-1, 1) → (-1, 2)
(2, 4) → (2, 8)
(-2, 4) → (-2, 8)

Example 1 (cont.)

Method 1: Sketch using Transformations

2. Translate the graph
   - Use the values of p and q to give the vertical and horizontal translation

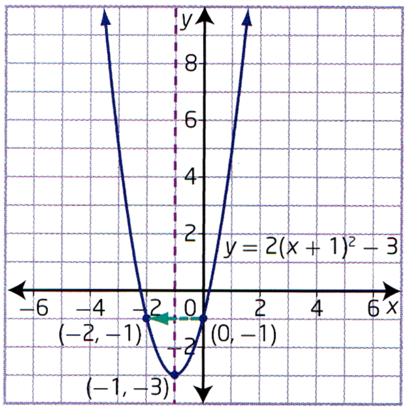


- $p = -1$, so the graph is translated one unit left
- $q = -2$, so the graph is translated two units down

Example 1 (cont.)

Method 2: Sketch using Points and Symmetry

- Plot the vertex, (-1, -3), and draw the axis of symmetry, x = -1
- determine the coordinates of at least 4 more points



a) Let x = 0

$$y = 2(0+1)^2 - 3$$

$$y = 2(1)^2 - 3$$

$$y = -1$$

The point is (0, -1) and there is a matching point across the axis of symmetry at (-2, -1)

Try: Determine the following for each function:

- the vertex
- the domain and range
- the direction of the opening
- the equation of the axis of symmetry

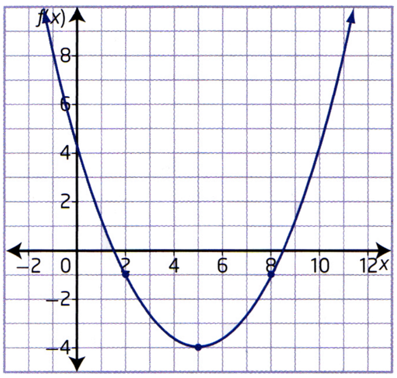
a) $y = \frac{1}{2}(x-2)^2 - 4$

b) $y = -3(x+1)^2 + 3$

Example 2

Determine a Quadratic Function Given Its Graph

Example: Determine a quadratic function in vertex form for the following graph.



Use Points and Substitution

1. Use the coordinates of the vertex and one other point:
   vertex (5, -4) and P (2, -1)
2. Substitute 5 and -4 for p and q into the vertex form of the equation.

$$f(x) = a(x-p)^2 + q$$

$$f(x) = a(x-5)^2 + (-4)$$

$$f(x) = a(x-5)^2 - 4$$

Example 2 cont.

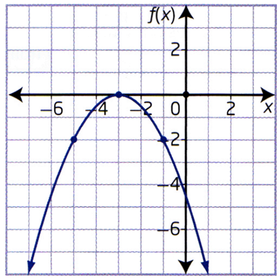
3. Solve the equation for a by substituting (2, -1) for x and y

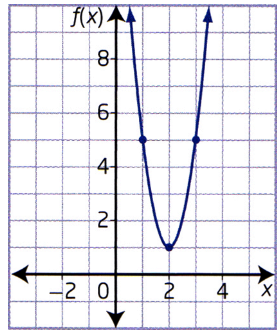
$$f(x) = a(x-5)^2 - 4$$
$$-1 = a(2-5)^2 - 4$$
$$-1 = a(-3)^2 - 4$$
$$-1 = a(-9) - 4$$
$$3 = 9a$$
$$\frac{1}{3} = a$$

4. Rewrite equation with using a ,p and q (but not x and y)

$$f(x) = a(x-p)^2 + q$$
$$f(x) = \frac{1}{3}(x-5)^2 - 4$$

Try: Determine a quadratic function in vertex form for the following graphs.

a) 



Example 3

Determine the Number of x-intercepts Using a and q

Example: Determine the number of x-intercepts for each quadratic function:

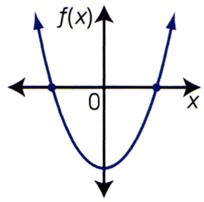
a) $f(x) = 0.8x^2 - 3$ b) $f(x) = 2(x-1)^2$ c) $f(x) = -3(x+2)^2 - 1$

You need to know:

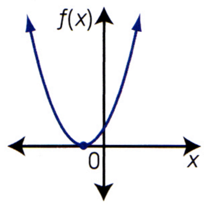
- the value of a to determine if the graph opens up or down
- the value of q to determine if the vertex is above, below or on the x-axis

Example 3 cont.

a) $f(x) = 0.8x^2 - 3$

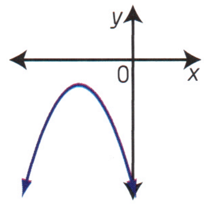
Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a > 0$ the graph opens upward	$q < 0$ the vertex is below the x-axis	 	2 crosses the x-axis *twice*, since it opens *upward* from a vertex *below* the x-axis

b) $f(x) = 2(x - 1)^2$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a > 0$ the graph opens upward	$q = 0$ the vertex is on the x-axis	 	1 touches the x-axis *once*, since the vertex is *on* the x-axis

Example 3 cont.

c) $f(x) = -3(x + 2)^2 - 1$

Value of a	Value of q	Visualize the Graph	Number of x-Intercepts
$a < 0$ the graph opens downward	$q < 0$ the vertex is below the x-axis		0 does not cross the x-axis, since it opens *down* from a vertex *below* the x-axis

Try: Determine the number of x -intercepts without graphing (use a and q)

a) $f(x) = 0.5x^2 - 7$

b) $f(x) = -2(x+1)^2$

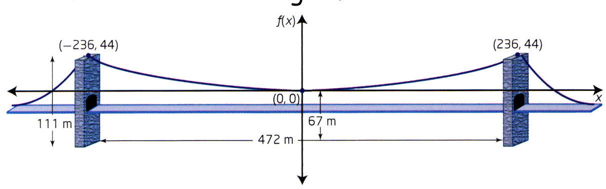
Example 4

Model Problems Using Quadratic Functions in Vertex Form

Example: The deck of a bridge is supported by 2 main cables attached to the tops of two towers. The cables are shaped like parabolas, with the lowest point approximately 67 m above the water. The towers are 111 m tall and 472 m apart.

a) Model the shape of the cables with a quadratic function in vertex form

1. Draw a labelled diagram



- place the vertex at the cables' low point & make it the origin
- put in the x and y-axes
- label the coordinates of the tops of the towers with respect to the vertex

Example 4 cont.

2. Determine the form of the equation.
   - Since a and q are both 0, the function will have the form $f(x) = ax^2$

3. Determine the equation.
   - Substitute the coordinates of one the towers into $f(x) = ax^2$ and solve for a

$$f(x) = ax^2$$

$$44 = a(236)^2$$

$$44 = 55696a$$

$$\frac{44}{55696} = \frac{55696}{55696}a$$

$$\frac{11}{13924} = a$$

3. Re-write the equation with a in place

$$f(x) = \frac{11}{13924}x^2$$

Example 4 cont.

b) Determine the height above the surface of the water of a point on the cables that is 90 m horizontally from one of the towers.

1. Determine the distance of the point from the vertex.
   - A point 90 m from one of the towers is 236-90, or 146 m horizontally from the vertex

2. Use the equation for the function to determine $f(146)$

$$f(x) = \frac{11}{13924}x^2$$

$$f(146) = \frac{11}{13924}(146)^2$$

$$f(146) = \frac{11}{13924}(21316)$$

$$f(146) = 16.839...$$

This is approximately 16.8 m above the low point in the cables which are 67 m above the water. The height above the water is 67 + 16.8 = 83.8 m

Try: A parabolic archway has a width of 280 cm and a height of 216 cm at its highest point.

a) Write a quadratic function in vertex form that models the shape of the archway

b) Determine the height of the archway at a point 50 cm from its outer edge.

Topic 2

Example 1

Quadratic Functions in Standard Form

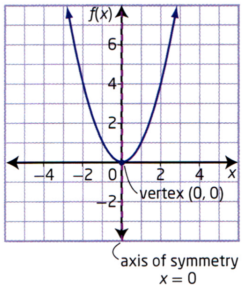
Identify Characteristics of a Quadratic Function in Standard Form

Example: For each graph of a quadratic function, identify:

- the direction of the opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and the y-intercept
- the domain and range

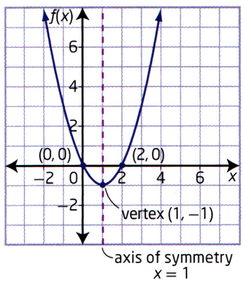
Example 1 cont.

a) $f(x) = x^2$



- opens upward
- vertex: (0, 0)
- minimum: $y = 0$ when $x = 0$
- axis of symmetry: $x = 0$
- y-intercept at (0, 0) and has a value of 0
- x-intercept at (0, 0) and has a value of 0
- domain: all real numbers or $\{x | x \in R\}$
- range: $y \geq 0$ or $\{y | y \geq 0, y \in R\}$

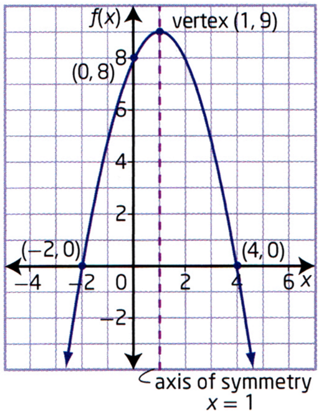
b) $f(x) = x^2 - 2x$



- opens upward
- vertex: (1, -1)
- minimum: $y = -1$ when $x = 1$
- axis of symmetry: $x = 1$
- y-intercept at (0, 0) and has a value of 0
- x-intercepts at (0, 0) & (2, 0) & have values of 0 & 2
- domain: all real numbers or $\{x | x \in R\}$
- range: $y \geq -1$ or $\{y | y \geq -1, y \in R\}$

Example 1 cont.

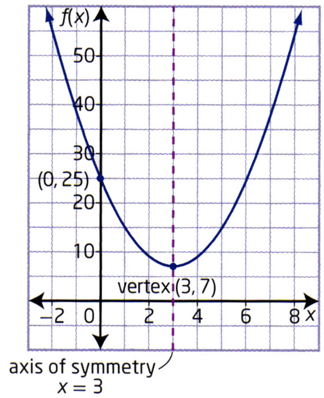
c) $f(x) = -x^2 + 2x + 8$



· opens downward
· vertex: (1, 9)
· maximum: y = 9 when x = 1
· axis of symmetry: x = 1
· y-intercept at (0, 8) and has a value of 8
· x-intercept at (-2, 0) & (4, 0) & have values of -2 & 4
· domain: all real numbers or $\{x | x \in R\}$
· range: $y \le 9$ or $\{y | y \le 9 | y \in R\}$

Example 1 cont.

d) $f(x) = 2x^2 - 12x + 25$

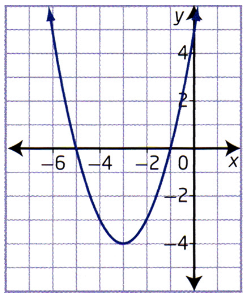


- opens upward
- vertex: (3, 7)
- minimum: y = 7 when x = 3
- axis of symmetry: x = 3
- y-intercept at (0, 25) and has a value of 25
- no x-intercepts
- domain: all real numbers or $\{x \mid x \in R\}$
- range: $y \geq 7$ or $\{y \mid y \leq 7, y \in R\}$

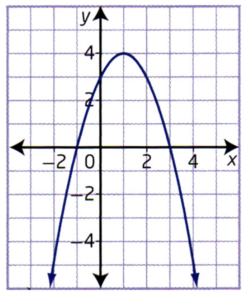
Try: For each graph of a quadratic function, identify:

- the direction of the opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and the y-intercept
- the domain and range

a) $f(x) = x^2 + 6x + 5$



b) $f(x) = -x^2 + 2x + 3$



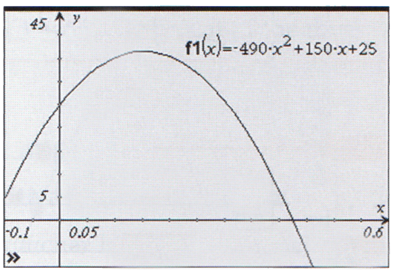
Example 2

Analysing a Quadratic Function

Example: A frog jumps into a pond. The height h, in cm, of the frog above the water is a function of time t, in seconds can be modeled by the function: $h(t) = -490t^2 + 150t + 25$

Answer the following:

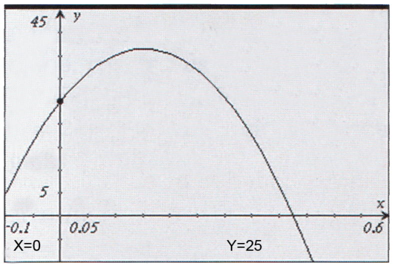
a) Graph the function - use a graphing calculator



1. Press $\boxed{\text{Y=}}$ and enter the function in Y = . Press $\boxed{\text{GRAPH}}$
2. You may have to adjust the size of the graph until the vertex and intercepts are visible. Press $\boxed{\text{WINDOW}}$ and change Xmin, Xmax, Ymin and Ymax as necessary and press $\boxed{\text{GRAPH}}$

Example 2 cont.

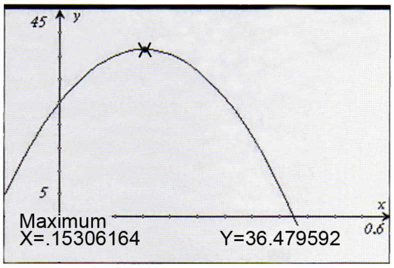
b) Find the y -intercept. What does it represent?



1. Press TRACE 0 ENTER
2. The y -intercept is (0, 25) or 25
   *note that this is the constant term in the equation
   $$h(t) = -490t^2 + 150t + 25$$
3. The y -intercept tells us that the height of the frog at the start of its jump was 25cm.

Example 2 cont.

c) What maximum height does the frog reach? When does it reach that height?



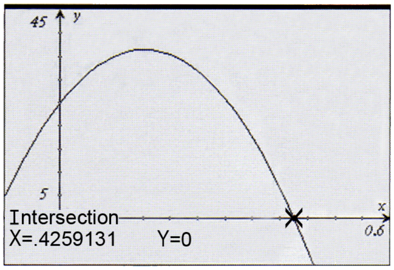
1. The vertex represents the time and height of the frog at its maximum point during the jump.

2. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ 4 (maximum) move the cursor to the left side of the maximum and press $\boxed{\text{ENTER}}$ then move the cursor to the right side of the maximum and press $\boxed{\text{ENTER}}$

3. The maximum occurs after about 0.2 s and the frog achieves a maximum height of about 36.5 cm

Example 2 cont.

d) When does the frog hit the water?



1. The positive x -intercept represents the time when the height is 0 cm, or when the frog hits the water.
2. Press Y= 0 GRAPH
3. Press 2nd TRACE 5 (intersection) Make sure the cursor is on the right side of the maximum and press ENTER three times.
4. The frog hits the water after approximately 0.2 s

Example 2 cont.

e) What are the domain and range in this situation?

1. The domain is the set of all possible values for the independent variable (time).

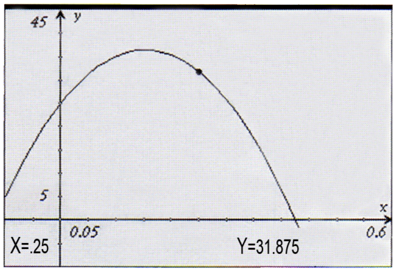
   $\{t \mid 0 \leq t \leq 0.4,\ t \in R\}$ Notice we start at 0 since negative values don't make sense for time in this case.

2. The range is the set of all possible values for the dependent variable (height).

   $\{h \mid 0 \leq h \leq 36.5,\ h \in R\}$ Notice we start at 0 since negative values don't make sense for height in this case.

Example 2 cont.

f) How high is the frog 0.25 s after it jumps?



1. The height of the frog after 0.25 s is the h-coordinate when t is 0.25.
2. Press TRACE 0.25 ENTER
3. The height of the frog is approximately 31.9 cm after 0.25 s
4. You can also calculate the height algebraically by substituting 0.25 for t in the function:

$$h(t) = -490t^2 + 150t + 25$$
$$h(0.25) = -490(0.25)^2 + 150(0.25) + 25$$
$$h(0.25) = -30.625 + 37.5 + 25$$
$$h(0.25) = 31.875$$

Try: A diver jumps from a 3 m springboard with an initial velocity of 6.8 m/s
Her height, h, in metres, above the water t seconds ofter leaving the board can be modelled by the function

$$h(t) = -4.9t^2 + 6.8t + 3$$

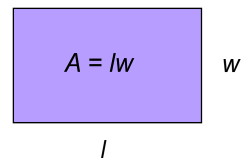
a) Graph the function
b) What does the y-intercept represent?
c) What is her maximum height? When does she reach that height?
d) How long until she hits the water?
e) What domain and range are appropriate for this situation?
f) What is the height of the diver 0.6 s after leaving the board?

Example 3

Write a Quadratic to Model a Situation

Example: A Rancher has 100 m of fencing available to build a rectangular corral.

a) Write a quadratic in standard form to represent the area of the corral.



1. Let l = length, w = width, a =
2. There are 2 formulas relating to rectangles:

   $A = lw$ and $P = 2l + 2w$

   Rearrange perimeter formula in terms of l

   $P = 2l + 2w$

   $100 = 2l + 2w$ or $2l + 2w = 100$

   $2l = 100 - 2w$

   $l = 50 - w$

Example 3 cont.

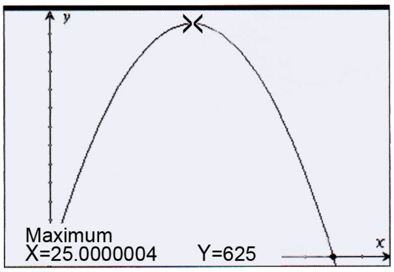
3. Substitute $l = 50 - w$ into $A = lw$

$A = lw$

$A = (50 - w)(w)$

$A = 50w - w^2$

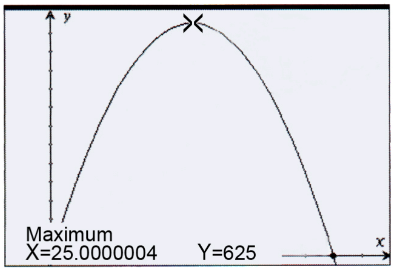
b) What are the coordinates of the vertex? What does the vertex represent in this situation?



1. Use the graphing calculator to graph $A = 50w - w^2$
2. Find the maximum as in part c) of example 2.
3. The vertex is at (25, 625). The y-intercept represents the maximum area of the rectangle and the x-intercept represents the width when this occurs.

Example 3 cont.

c) Sketch the graph for the function from part a)



Example 3 cont.

d) Determine the domain and range for this situation.

Negative lengths, widths and areas don't have any meaning in this situation, so the domain and range are restricted.

1. The width is any real number from 0 to 50

   $\{w \mid 0 \leq w \leq 50,\ w \in R\}$

2. The area is any real number from 0 to 625

   $\{a \mid 0 \leq a \leq 625,\ a \in R\}$

e) Identify any assumptions you made in modeling this situation mathematically.

1. The equation written in part a) assumes that all the fencing will be used
2. It also assumes that any width or length is possible

Try: At a children's music festival, the organizers are roping off a rectangular area for stroller parking. There is 160 m of rope available to create the perimeter.

a) Write a quadratic in standard form to represent the area for stroller parking
b) What are the coordinates of the vertex? What does the vertex represent in this situation?
c) Sketch the graph for the function from a)
d) Determine the domain and range for this situation.
e) Identify any assumptions you made.